Lecture 11

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1 Examples of bases

Last time we studied bases of vector spaces. Today we're going to give some examples of bases.

Example 1.1. *Consider the vector space $\mathbb{P}_2$ — the space of polynomials with degree less than or equal to 2. Let's consider the following 3 vectors in this vector space:*

$$u_1 = t^2 + 1, \quad u_2 = t + 1, \quad u_3 = t - 1.$$

Let's determine whether it is a basis or not. We have to check 2 conditions:

Spanning set *To check that these vectors form a spanning set for $\mathbb{P}_2$ we should take arbitrary vector from $\mathbb{P}_2$ and try to express it as a linear combination of the vectors from the basis. Let's take arbitrary polynomial $at^2 + bt + c$:*

$$at^2 + bt + c = x(t^2 + 1) + y(t + 1) + z(t - 1) = xt^2 + (y + z)t + (x + y - z).$$

So, we can see that this is equivalent to the following system of linear equations, which we will try to solve:

$$\begin{cases} x & & & = a \\ & y + z & & = b \\ x + y - z & & & = c \end{cases} \overset{\text{subtract the 1st eq. from the 3rd one}}{\rightsquigarrow}$$

$$\begin{cases} x & = a \\ y + z & = b \\ y - z & = c - a \end{cases} \overset{\text{subtract the 2nd eq. from the 3rd one}}{\rightsquigarrow}$$

$$\begin{cases} x & = a \\ y + z & = b \\ -2z & = c - a - b \end{cases}$$

So, we see that $z = \frac{1}{2}(a+b-c)$, $y = \frac{1}{2}(b+c-a)$, *and* $x = a$. *So, we got the expression for arbitrary polynomial as a linear combination of given:*

$$at^2 + bt + c = a(t^2+1) + \frac{1}{2}(b+c-a)(t+1) + \frac{1}{2}(a+b-c)(t-1).$$

So, this system is a spanning set.

Linear independence *To check that these vectors are linearly independent we form a linear combination which is equal to 0:*

$$x(t^2+1) + y(t+1) + z(t-1) = 0 \Leftrightarrow xt^2 + (y+z)t + (x+y-z) = 0.$$

This is equivalent to the following linear system:

$$\begin{cases} x & & & & & = & 0 \\ & & y & + & z & = & 0 \\ x & + & y & - & z & = & 0 \end{cases} \overset{\text{subtract the 1st eq. from the 3rd one}}{\rightsquigarrow} \begin{cases} x & & & & & = & 0 \\ & & y & + & z & = & 0 \\ & & y & - & z & = & 0 \end{cases} \overset{\text{subtract the 2nd eq. from the 3rd one}}{\rightsquigarrow} \begin{cases} x & & & & & = & 0 \\ & & y & + & z & = & 0 \\ & & & - & 2z & = & 0 \end{cases}$$

So, we see that the only solution for this system is $x = 0$, $y = 0$, *and* $z = 0$. *Thus, these vectors are linearly independent.*

So, since both properties hold for this system of vectors, we deduce that this system is a basis.

Example 1.2. *Consider the vector space* $\mathbb{P}_2$ *— the space of polynomials with degree less than or equal to 2. Let's consider the following 3 vectors in this vector space:*

$$u_1 = t^2 + t + 2, \quad u_2 = t^2 + 1, \quad u_3 = t + 1.$$

Let's determine whether it is a basis or not. We have to check 2 conditions:

Spanning set *To check that these vectors form a spanning set for* $\mathbb{P}_2$ *we should take arbitrary vector from* $\mathbb{P}_2$ *and try to express it as a linear combination of the vectors from the basis. Let's take arbitrary polynomial* $at^2 + bt + c$:

$$at^2 + bt + c = x(t^2+t+2) + y(t^2+1) + z(t+1) = (x+y)t^2 + (x+z)t + (2x+y+z).$$

So, we can see that this is equivalent to the following system of linear equations, which we will try to solve:

$$\begin{cases} x + y = a \\ x + z = b \\ 2x + y + z = c \end{cases} \underset{\text{and subtract the 1st eq. from the 2nd one}}{\overset{\text{subtract the 1st eq. mult. by 2 from the 3rd one,}}{\rightsquigarrow}} \begin{cases} x + y = a \\ -y + z = b - a \\ -y + z = c - 2a \end{cases} \overset{\text{subtract the 2nd eq. from the 3rd one}}{\rightsquigarrow} \begin{cases} x + y = a \\ -y + z = b \\ 0 = c - a - b \end{cases}$$

So, we see that this system has no solution if $c - a - b \neq 0$. *For example, if* $a = 1$, $b = 1$, $c = 1$, *then* $c - a - b = 1 - 1 - 1 = -1 \neq 0$, *so the vector* $at^2 + bt + c = t^2 + t + 1$ *can not be expressed as a linear combination of the given vectors. So we deduce, that this system of vectors is not a basis.*

Actually, here we can stop, and do not check the linear independence — we know, that it is not a basis already!!! But we will show how to check that these vectors are linearly dependent.

Linear independence *To find whether these vectors are linearly independent or not we form a linear combination which is equal to 0:*

$$x(t^2 + t + 2) + y(t^2 + 1) + z(t + 1) = 0 \Leftrightarrow (x + y)t^2 + (x + z)t + (2x + y + z) = 0.$$

This is equivalent to the following linear system:

$$\begin{cases} x + y = 0 \\ x + z = 0 \\ 2x + y + z = 0 \end{cases} \underset{\text{and subtract the 1st eq. from the 2nd one}}{\overset{\text{subtract the 1st eq. mult. by 2 from the 3rd one,}}{\rightsquigarrow}} \begin{cases} x + y = 0 \\ -y + z = 0 \\ -y + z = 0 \end{cases} \overset{\text{subtract the 2nd eq. from the 3rd one}}{\rightsquigarrow} \begin{cases} x + y = 0 \\ -y + z = 0 \\ 0 = 0 \end{cases}$$

So we see that this system has nonzero solution, for example $(1, -1, -1)$. *So, the linear combination with these coefficients is non trivial and is equal to 0:*

$$1(t^2 + t + 2) - 1(t^2 + 1) - 1(t + 1) = 0$$

Thus these vectors are linearly dependent.

2 Dimension

Now we'll state the following theorem about linear dependence.

Theorem 2.1 (Main lemma about linear dependence). *Let $u_1, u_2, \dots, u_n$ is a basis for vector space V. Let $m > n$. Then any m vectors from V are linearly dependent.*

Example 2.2. *Vectors $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$ form a basis for $\mathbb{R}^2$. So, any 3 vectors from $\mathbb{R}^2$ are linearly dependent. For example we can say that*

$$v_1 = \begin{pmatrix}2\\4\end{pmatrix}, \quad v_2 = \begin{pmatrix}5\\2\end{pmatrix}, \quad v_3 = \begin{pmatrix}0\\1\end{pmatrix}$$

are linearly dependent without finding a nontrivial linear combination of them.

The following corollary is one of the main results in linear algebra.

Corollary 2.3. *All bases of the given vector space V have the same number of vectors.*

Definition 2.4. *The number of vectors in basis of V is called the* ***dimension*** *of V. It is denoted by $\dim V$.*

Example 2.5. *The space $\mathbb{R}^2$ has 2 vectors in its basis, so $\dim \mathbb{R}^2 = 2$.*

Example 2.6. *The space $\mathbb{P}_2$ of polynomials of degree less than or equal to 2 has dimension equal to 3, since it has a basis of 3 vectors: $u_1(t) = t^2$, $u_2(t) = t$, and $u_3(t) = 1$. So, $\dim \mathbb{P}_2 = 3$.*

Now we are ready to give the proofs of these main results.

Proof of the Main Lemma about linear dependence. Let we have m vectors in the V: v_1, v_2, $\dots$, v_m, and $m > n$, where n is dimension of V. Vectors $u_1, u_2, \dots, u_n$ form a basis for V, so we can express vectors v_i's as linear combinations of u_i's:

$$\begin{aligned}
v_1 &= a_{11}u_1 + a_{12}u_2 + \cdots + a_{1n}u_n \\
v_2 &= a_{21}u_1 + a_{22}u_2 + \cdots + a_{2n}u_n \\
&\cdots \\
v_m &= a_{m1}u_1 + a_{m2}u_2 + \cdots + a_{mn}u_n
\end{aligned}$$

Let's form a linear combination of v_i's which is equal to zero, and prove that it may be non-trivial — then it will be proved that v_i's are linearly dependent.

$$\begin{aligned}\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_m v_m = &\lambda_1(a_{11}u_1 + a_{12}u_2 + \cdots + a_{1n}u_n)\\ &+ \lambda_2(a_{21}u_1 + a_{22}u_2 + \cdots + a_{2n}u_n)\\ &+ \cdots\\ &+ \lambda_m(a_{m1}u_1 + a_{m2}u_2 + \cdots + a_{mn}u_n)\end{aligned}$$

By rearranging terms, we write that the same linear combination is equal to

$$\begin{aligned}\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_m v_m = &u_1(\lambda_1 a_{11} + \lambda_2 a_{21} + \cdots + \lambda_m a_{m1})\\ &+ u_2(\lambda_1 a_{12} + \lambda_2 a_{22} + \cdots + \lambda_m a_{m2})\\ &+ \cdots\\ &+ u_n(\lambda_1 a_{1n} + \lambda_2 a_{2n} + \cdots + \lambda_m a_{mn})\end{aligned}$$

In order for it to be equal to $\mathbf{0}$, we will write that coefficients are equal to 0 (since u_i's are independent). We'll have the system of linear equations:

$$\begin{cases}\lambda_1 a_{11} & + & \lambda_2 a_{21} & + & \cdots & + & \lambda_m a_{m1} & = & 0\\ \lambda_1 a_{12} & + & \lambda_2 a_{22} & + & \cdots & + & \lambda_m a_{m2} & = & 0\\ \multicolumn{9}{c}{\dotfill}\\ \lambda_1 a_{1n} & + & \lambda_2 a_{2n} & + & \cdots & + & \lambda_m a_{mn} & = & 0\end{cases}$$

This is a homogeneous system, and the number of equations is n, the number of variables is m, so the number of equations is less then the number of variables (since $n < m$). So, it has non-trivial solution — there exist λ_i's not all equal to 0, such that linear combination of v_i's is equal to $\mathbf{0}$. So, v_i's are linearly dependent. □

Proof of the Corollary 2.3. Let we have 2 bases with different numbers of vectors, say m in the first basis, and n in the second one. Let $m > n$. But by the previous theorem any m vectors are linearly dependent. But they are in basis, so they should be independent! Contradiction proves the corollary. □